A Schiffer-type problem for annular domains

joint work with A. Enciso, A. J. Fernández and P. Sicbaldi

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Outline

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The Pompeiu problem

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function and $\Omega \subset \mathbb{R}^N$ a bounded domain. The Pompeiu problem consists in recovering f from the values:

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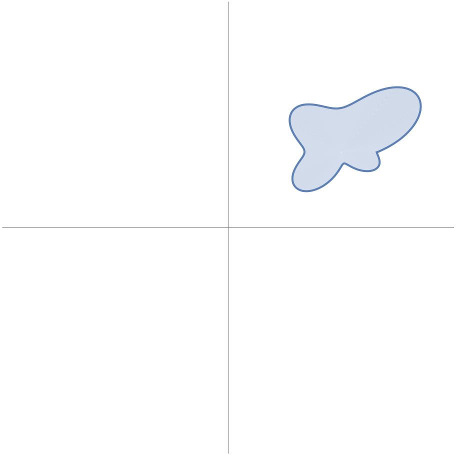
where $\mathcal{R}$ is any rigid motion in $\mathbb{R}^N$.

We say that Ω satisfies *the Pompeiu property* if $f = 0$ is the unique function f

$$\int_{\mathcal{R}(\Omega)} f(x)\,dx = 0$$

for any rigid motion $\mathcal{R}$.

The Pompeiu problem



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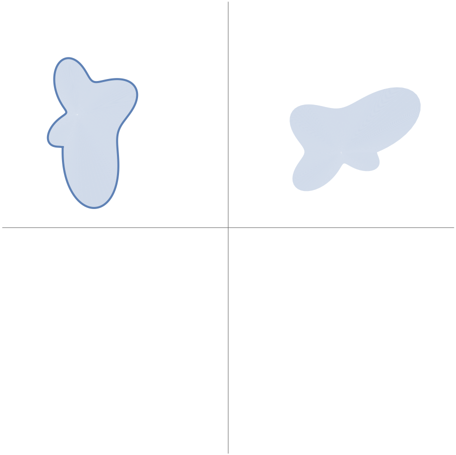

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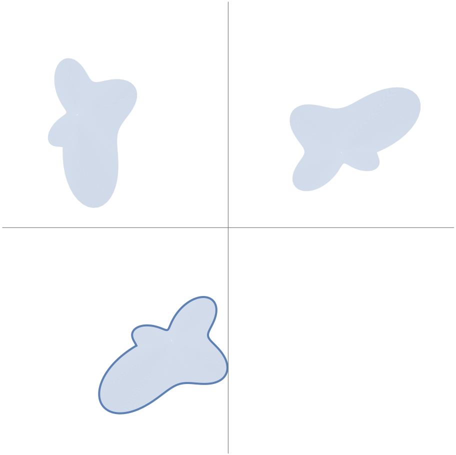

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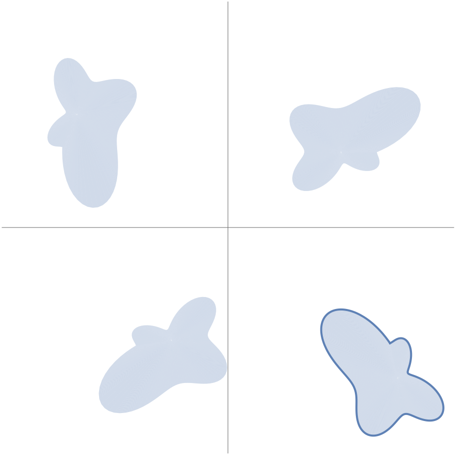

Ellipses and polygons satisfy the Pompeiu property (Brown, Schreiber & Taylor '73).

Balls do not satisfy the Pompeiu property!

Let Ω be any ball of radius R, and $\mu \neq 0$ a radial Neumann eigenvalue; then,

$$\begin{cases} -\Delta u - \mu u = 0 & \text{in } \Omega, \\ u = c & \text{on } \partial\Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $w(x) = (\mu c)^{-1}(u(x) - c)$.

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We now take $f : \mathbb{R}^2 \to \mathbb{R}$ such that $-\Delta f = \mu f$. For instance, $f(x) = \sin(\sqrt{\mu} x_1)$, or $f(x) = J_{\frac{N-2}{2}}(\sqrt{\mu}|x|)$.

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An overdetermined elliptic problem

Observe that the above argument works as long as there exists a solution to the problem:

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Theorem (Williams '76)

If $\partial\Omega$ is homeomorphic to $\mathbb{S}^{N-1}$ and Ω does not satisfy the Pompeiu property, then there exists a solution to (S).

The topological assumption is necessary!

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2. If $\partial\Omega$ is connected and there exists a diverging sequence μ_k for which (S) is solvable, then Ω is a ball (Berenstein '80, Berenstein-Yang '87).

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1. If Ω is C^1 and there exists a solution to (S), then Ω is analytic (Kinderlehrer-Nirenberg '77).
2. If $\partial\Omega$ is connected and there exists a diverging sequence μ_k for which (S) is solvable, then Ω is a ball (Berenstein '80, Berenstein-Yang '87).
3. If $N = 2$, $\partial\Omega$ is connected and (S) admits a solution for $\mu \leq \mu_6$, then Ω is a disk (Avilés '86, Deng '12).
   Here we are denoting the Neumann eigenvalues as:

$$0 = \mu_0 < \mu_1 \leq \mu_2 \dots$$

Outline

A related question

In this talk we are interested in the following question:

Question

Let $\Omega \subset \mathbb{R}^2$ a domain, and let us denote Γ_i the connected components of $\partial\Omega$. Assume that there exists a solution to the problem:

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ u = c_i & \text{on } \Gamma_i, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \mu > 0. \tag{Q}$$

Is it true that Ω is a disk or an annulus?

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Observe that here, as in problem (S), we have that $\nabla u = 0$ on $\partial\Omega$.

Rigidity results

This problem shares many features with the Schiffer conjecture. To start with, if Ω is C^1 and there exists a solution to (Q), then Ω is analytic (Kinderlehrer-Nirenberg '77).

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Proposition

If there exists a diverging sequence μ_k for which (Q) *is solvable, then Ω is a disk or an annulus.*

The proof uses the same ideas of Berenstein together with a unique continuation argument.

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Assume that (Q) *admits a solution. Then:*

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i) *If* $\mu \leq \mu_4$, *then* Ω *is a disk or an annulus.*

ii) *If* $\mu \leq \mu_5$ *and* $\partial\Omega$ *has exactly two connected components, then* Ω *is an annulus.*

The proof uses some of the ideas of Avilés and Deng, together with a comparison result between Dirichlet and Neumann eigenvalues (Friendlander '95, Filonov '05).

In case ii) we also need to make use of the result of Reichel for overdetermined problems in annuli.

Main result

The answer to the question is NO!

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Theorem

There exist nonradial domains $\Omega = \Omega_0 \setminus \Omega_1$, *where* $\Omega_i \subset \mathbb{R}^2$ *are bounded smooth and simply connected,* $\overline{\Omega_1} \subset \Omega_0$, *such that the problem:*

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ u = c_i & \text{on } \Gamma_i, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a solution. Here $\mu > 0$ *and* $\Gamma_i = \partial\Omega_i$.

Connection to stationary Euler flows in 2D

The stationary Euler equations are:

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \mathbb{R}^2\,, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^2\,. \end{cases} \tag{E}$$

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In particular, one can take compactly supported radial functions.

It is not known if there are compactly supported C^1 solutions to (E) with noncircular streamlines.

Connection to stationary Euler flows in 2D

Take u as given by our main theorem. Then, define

$$v = \begin{cases} (\nabla u)^{\perp} & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 \backslash \Omega, \end{cases} \qquad p = \begin{cases} \mu \left(\frac{c_0^2}{2} - \frac{c_1^2}{2} \right) & \text{in } \overline{\Omega_1}, \\ -\frac{1}{2}(|\nabla u|^2 + \mu(u^2 - c_0^2)) & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega_0 \,. \end{cases}$$

Then (v, p) are continuous weak solution to (E) with non-circular streamlines and compact support.

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Another compactly supported continuous weak solution of (E) has been given by Gómez-Serrano, Park and Shi. Their example is of vortex-patch type and the streamfunction does not solve any elliptic PDE.

Outline

Sketch of the proof

We consider a 1-parametric family of annuli $A(0; a, 1)$, where $a \in (0, 1)$.

We denote $\mu_{0,k}(a)$ the k-th radial eigenvalue, with $\mu_{0,0}(a) = 0$.
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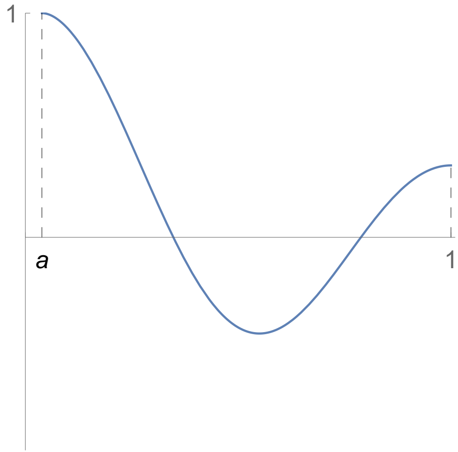
Our aim is to show that, for some $a^* \in (0, 1)$, there exists a branch of nonradial solutions of (Q) bifurcating from $A(0; a^*, 1)$ and ψ_{a^*}.

We cannot perform this argument from $\mu_{0,1}(a)$: in this case, the eigenfunction attains its minimum and maximum on the boundary. Hence also a bifurcating solution would do so. But in this case it is known that Ω has to be an annulus (Reichel '95).

Sketch of the proof



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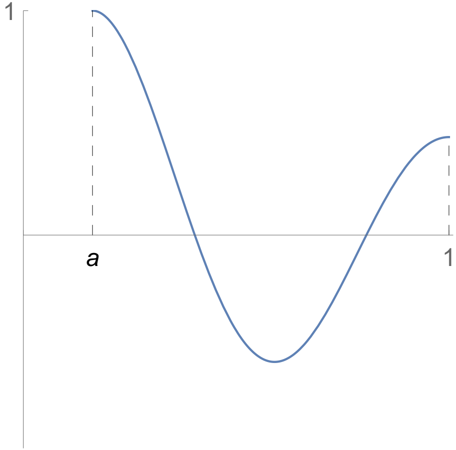

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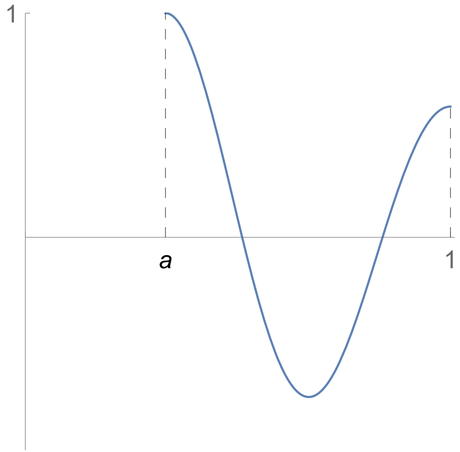

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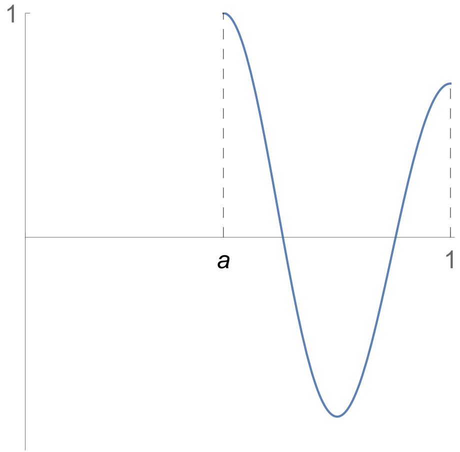

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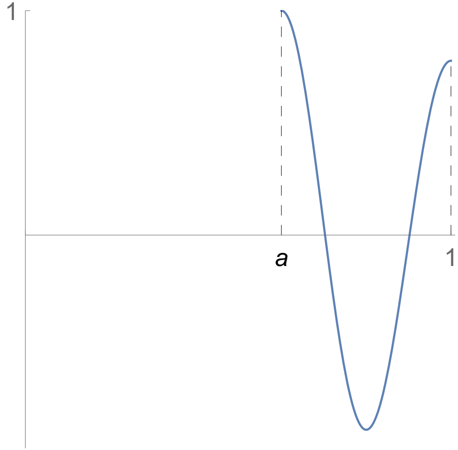

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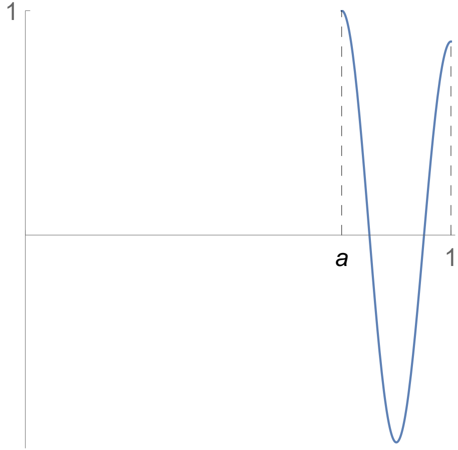

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A direct approach

In polar coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^1$, we define:

$$\Omega_a^{b,B} = \{(r,\theta) : \; a + b(\theta) < r < 1 + B(\theta)\}.$$

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For small $(b, B) \in C^{k,\alpha}(\mathbb{S}^1)^2$, we can solve:

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega_a^{b,B}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_a^{b,B}. \end{cases}$$

Then we can define $F(b, B)$ and $\tilde{F}(b, B) \in C^{k,\alpha}(\mathbb{S}^1)^2$,

$$F(b,B)(\theta) = (u(a + b(\theta), \theta)), u(1 + B(\theta), \theta)),$$

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The zeroes of $\tilde{F}$ are the solutions to our problem. Clearly, $\tilde{F}(0, 0) = 0$ for any $a \in (0, 1)$.

A direct approach

It turns out that:

$$D\tilde{F}_{(0,0)}(\hat{b},\hat{B})(\theta) = (c_1(a)\Psi(a,\theta), c_2(a)\Psi(1,\theta)),$$

where $c_i(a) \neq 0$ are constants and $\Psi = \Psi_{\hat{b},\hat{B}}$ solves:

$$\begin{cases} -\Delta\Psi = \mu\Psi & a < r < 1, \\ \partial_\nu u = \hat{b}(\theta) & r = a, \\ \partial_\nu u = \hat{B}(\theta) & r = 1, \end{cases}$$

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However, it turns out that:

$$D\tilde{F}_{(0,0)} : C^{k,\alpha}(\mathbb{S}^1)^2 \to C^{k+1,\alpha}(\mathbb{S}^1)^2.$$

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However, a new approach that avoids this problem has been recently found by Fall, Minlend and Weth.

The functional framework

Denote by $\Omega_{1/2} = A(0; 1/2, 1)$, and $\Phi_a^{b,B}$ a diffeomorphism:

$$\Phi_a^{b,B} : \Omega_{1/2} \to \Omega_a^{b,B}, \;\; L_a^{b,B} = (\Phi_a^{b,B})^* \Delta + \mu_{0,2}(a) Id.$$

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For $b = B = 0$ we have the 2nd Neumann eigenfunction ψ_a, and $\bar{\psi}_a = \psi_a \circ \Phi_a^{0,0}$.
We also define the function spaces:

$$\mathcal{X}^{k,\alpha} = \{u \in C^{k,\alpha}(\Omega_{1/2}) : \;\; \partial_r u \in C^{k,\alpha}(\Omega_{1/2})\},$$

The functional framework

Denote by $\Omega_{1/2} = A(0; 1/2, 1)$, and $\Phi_a^{b,B}$ a diffeomorphism:

$$\Phi_a^{b,B} : \Omega_{1/2} \to \Omega_a^{b,B},\ \ L_a^{b,B} = (\Phi_a^{b,B})^* \Delta + \mu_{0,2}(a) Id.$$

For $b = B = 0$ we have the 2nd Neumann eigenfunction ψ_a, and $\bar{\psi}_a = \psi_a \circ \Phi_a^{0,0}$.
We also define the function spaces:

$$\mathcal{X}^{k,\alpha} = \{u \in C^{k,\alpha}(\Omega_{1/2}) :\ \partial_r u \in C^{k,\alpha}(\Omega_{1/2})\},$$

$$\mathcal{X}_D^{k,\alpha} = \{u \in \mathcal{X}^{k,\alpha} :\ u = 0 \text{ on } \partial\Omega_{1/2}\},$$

$$\mathcal{X}_{DN}^{k,\alpha} = \{u \in \mathcal{X}_D^{k,\alpha} :\ \partial_r u = 0 \text{ on } \partial\Omega_{1/2}\}.$$

The nonlinear operator

We define the operator:

$$G_a(v) = L_a^{b_v,B_v}[\bar{\psi}_a + w_v],$$

where v is in a neighborhood of 0 in $\mathcal{X}_D^{2,\alpha}$,

$$b_v(\theta) = c_1(a)\partial_r v\left(\tfrac{1}{2}, \theta\right),$$
$$B_v(\theta) = c_2(a)\partial_r v(1, \theta),$$

and $w_v \in \mathcal{X}_{DN}^{2,\alpha}$ is defined as:

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and $w_v \in \mathcal{X}_{DN}^{2,\alpha}$ is defined as:

$$w_v(r, \theta) = v(r, \theta) + \frac{\bar{\psi}_a'(r)}{2(1-a)}\Big[2(1-r)b_v(\theta) + (2r-1)B_v(\theta)\Big].$$

The linearization

Proposition

The map G is defined in a neighborhood of 0 *in* $\mathcal{X}_D^{2,\alpha}$ *and takes values in* $\mathcal{Y}$*, defined as:*

$$\mathcal{Y} = C^{1,\alpha}(\overline{\Omega}_{\frac{1}{2}}) + \mathcal{X}_D^{0,\alpha},$$

endowed with the norm

$$\|u\|_{\mathcal{Y}} = \inf\left\{\|u_1\|_{C^{1,\alpha}} + \|u_2\|_{\mathcal{X}^{0,\alpha}} : u_1 \in C^{1,\alpha},\ u_2 \in \mathcal{X}_D^{0,\alpha},\ u = u_1 + u_2\right\}.$$

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Moreover, $DG_a(0)(v) = L_a^{0,0}(v)$ is a Fredholm operator of index 0 from $\mathcal{X}_D^{2,\alpha}$ to $\mathcal{Y}$.
This operator becomes degenerate if μ is a Dirichlet eigenvalue.

Eigenvalues' crossing

Recall that $\mu_{0,2}(a)$ is the second Neumann eigenvalue which is radially symmetric in $A(0; a, 1)$.
We now denote $\lambda_{l,0}(a)$ the first Dirichlet eigenvalue for functions of mode l, that is, $\psi(r, \theta) = \phi(r)\cos(l\theta)$.

Proposition

The following asymptotics hold:

1. *If a is close to 1, then $\mu_{0,2}(a) > \lambda_{l,0}(a)$.*
2. *If a is close to 0 and $l \geq 4$, then $\mu_{0,2}(a) < \lambda_{l,0}(a)$.*

Then, there exists some $a^* \in (0, 1)$ for which $\mu_{0,2}(a^*) = \lambda_{l,0}(a^*)$.

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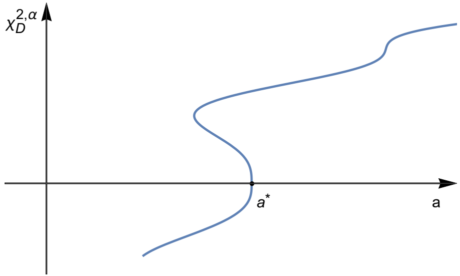
To exclude resonances with lower modes we restrict ourselves to l-symmetric functions, $l \geq 4$. Moreover this allows us to rule out the invariance by translations.

Bifurcation

The degeneracy of $DG_a(0)$ is necessary to obtain bifurcation.



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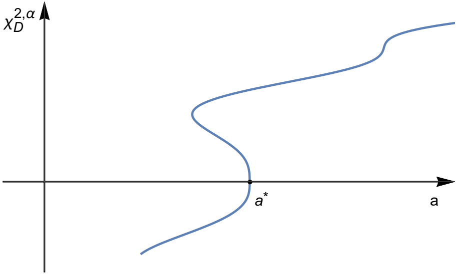


By the classical Crandall-Rabinowitz theorem, bifurcation occurs if the transversality condition holds, that here reduces to:

$$\mu'_{0,2}(a^*) > \lambda'_{l,0}(a^*).$$

We have been able to prove this only for large l.

Theorem

Let $l \geq l_0$ sufficiently large. There exist some $\varepsilon > 0$ and a continuously differentiable curve

$$(-\varepsilon, \varepsilon) \ni s \mapsto \big\{(a(s), b_s, B_s) \subset (0,1) \times C^{2,\alpha}(\mathbb{S}^1) \times C^{2,\alpha}(\mathbb{S}^1),$$

$$(a(0), b_0, B_0) = (a^*, 0, 0)\big\}, \text{ and}$$

$$b_s(\theta) = s\,c\cos(l\theta) + o(s)\,,\;\; B_s(\theta) = s\,C\cos(l\theta) + o(s),\;\; C > 0 > c,$$

such that the following problem admits a solution:

$$\begin{cases} \Delta u_s + \mu_{0,2}(a(s))u_s = 0 & \text{in } \Omega_{a(s)}^{b_s,B_s}\,, \\ \nabla u_s = 0 & \text{on } \partial\Omega_{a(s)}^{b_s,B_s}. \end{cases}$$

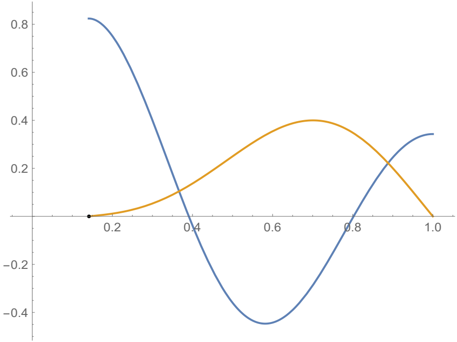
A posteriori, the domain and the solution are analytic.

Numerical approximation

For specific values of l one can give numerical approximations by using Mathematica. For instance, if $l = 4$,

$$a^* = 0,140989\ldots, \quad \mu_{0,2}(a^*) = \lambda_{4,0}(a^*) = 57,5851\cdots = \mu_{18},$$

$$\mu'_{0,2}(a^*) = 105,971\ldots, \; \lambda'_{4,0}(a^*) = 0,12067\ldots$$



Thank you for your attention!